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Trajectory and Control Optimization for Flexible Space Robots

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This paper is concerned with a flexible space robot whose mission is to ferry some payload in space and to dock smoothly with an orbiting target. Using a perturbation approach, the problem is divided into one for the rigid-body maneuvering of the robot and one for the feedback control of elastic vibrations and perturbations from the rigid-body maneuvering motions. The procedure involves optimization of the robot trajectory, inverse dynamics for the determination of the maneuvering forces and torques, and the use of the linear quadratic regulator (LQR) theory for a time-varying system for the determination of an optimal control law for the elastic motions and the perturbation from the rigid-body maneuvering motions. A numerical example demonstrates the approach.

I. Introduction

SPACE robots are likely to play an increasingly important role in space missions. Indeed, space robots can be used for the collection of space debris, recovery of spacecraft stranded in a useless orbit, the repair of orbiting spacecraft, the construction of a space station in orbit, and servicing the space station while in operation. The basic function of the space robot is to move payloads from one position to another. In contrast with ordinary manipulators, however, space robots must possess the capability of traveling in space and docking with other spacecraft.[1]

There are two significant differences between industrial robots in current use and space robots. In the first place, industrial robots are mounted on a fixed base, whereas space robots are mounted on space platforms capable of translations and rotations. The fact that the base of the space robot is free to move in space brings about problems[2,3] not encountered in industrial robots. The second significant difference is that space robots must be very light, and hence very flexible, unlike industrial robots characterized by very bulky and stiff arms. Indeed, the flexibility of the robot arms causes elastic vibration, which tends to affect adversely the performance of the end effector. This in turn causes problems in modeling and control design.[4]

Quite recently, extensive research has been done for space-based robots with free-flying characteristics. Vafa and Dubowsky[5] developed a new concept, referred to as a virtual manipulator (VM), to represent the free-flying manipulator system. Manipulator path planning and base attitude adjustment can be carried out by cyclic joint motions. Alexander and Cannon[6] presented an extended operational-space control algorithm to calculate appropriate joint torques so as to permit the end effector to track a desired path while allowing for the free dynamic response of the base vehicle. The main thrust of the research on free-flying space robots is to control the system by merely maneuvering the robot arms. One can realize the desired end-effector trajectory and desired base attitude simultaneously[7] or realize the desired joint angles and desired base attitude simultaneously[8] by actuating joint angles without vehicle attitude control.

The research mentioned above is based on the assumptions that there are no external forces or torques, so that the system linear or angular momentum is constant. These assumptions hold only when the robot base is not controlled. However, at times the robot base must be controlled. The advantage of controlling the robot base attitude is that it allows for a much simpler inverse kinematics relation between robot manipulation and robot configuration variables. The trajectory associated with the robot configuration can be optimized or designed using a great variety of performance indices or specific requirements.[9] For example, trajectories can be planned so as to minimize the base reactions and limit the end-effector accelerations and jerks to some values,[10] whereas the robot redundancy is being treated by a local optimization method.[9]

A number of investigations are concerned with the maneuvering of rigid fixed-base manipulators. A notable exception is a review paper[11] in which the arms are flexible and their vibration is actively controlled. The main conclusion is that no control scheme seems to have advantages over the other.

The approach presented in this paper is based on the assumption that the attitude is controlled so that the robot base maintains a fixed orientation during maneuvering. The inverse kinematics is the same as that of a ground-based robot, except that the base undergoes translational motion. The problem of robot redundancy is treated by the global optimization method, which reduces it to a nonlinear two-point boundary value problem. To avoid numerical difficulties in finding a global minimum solution to the problem,[12] a homotopy algorithm guaranteeing convergence to the global optimal trajectory[13] is used.

Most research on space robots is confined to rigid-body models. In contrast, this paper is concerned with the problem of accurate and smooth docking of a flexible space robot. The mathematical model consists of a rigid platform serving as a base, two hinge-connected flexible arms, and a rigid end effector/payload (Fig. 1). The complexity of inverse kinematics is reduced by assuming that the space robot attitude is controlled. The translational degrees of freedom permit optimality in trajectory planning. The equations for the coupled motions of the base, flexible robot arms and end effector/payload are derived by means of Lagrange's equations. Using a perturbation approach, the problem is separated into one for the docking maneuver of the space robot regarded as rigid and another for the suppression of the elastic vibrations and perturbations in the rigid-body motions. The controls for the first problem are obtained by inverse dynamics. On the other hand, the controls for the second problem are optimal feedback controls obtained by the LQR theory for a time-varying system. A numerical example demonstrates the approach.

II. Robot Redundancy and Global Optimization Method

The problem of interest here is the determination of the state vector defining the motion of the robot under the assumption that the tip trajectory of the end effector/payload is given. We denote by $\boldsymbol{r} \in R^m$ the manipulation variables vector defining the tip trajectory and by $\boldsymbol{q} \in R^n$ the vector of generalized coordinates defining the robot configuration vector. For a redundant robot, $m < n$, one tip trajectory corresponds to an infinity of state trajectories.

Let us assume that the kinematic relation between $\boldsymbol{r}$ and $\boldsymbol{q}$ is given by

$$\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q}) \tag{1}$$

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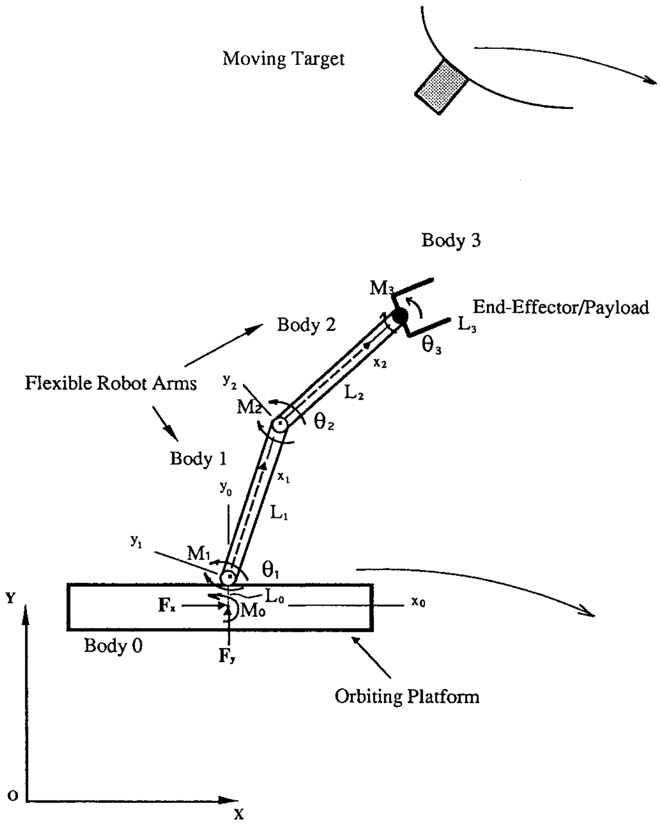

Fig. 1 Flexible space robot.

Taking the derivative of Eq. (1) with respect to time, we obtain

$$\dot{r} = J(q)\dot{q} \tag{2}$$

where

$$J(q) = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \cdots & \frac{\partial f_2}{\partial q_n} \\ \hdashline \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} = \left[\frac{\partial f^T}{\partial q}\right]^T \tag{3}$$

is the Jacobian matrix.

The object is to determine a state trajectory corresponding to a given tip trajectory. To this end, we invert Eq. (2) and write the solution in the form

$$\dot{q} = J^{\dagger}\dot{r}(t) + (I_n - J^{\dagger}J)y \tag{4}$$

where

$$J^{\dagger} = J^T(JJ^T)^{-1} \tag{5}$$

is the pseudoinverse of the matrix J, I_n is the $n \times n$ identity matrix, and y is an arbitrary n-vector. Note that the first term on the right side of Eq. (4) represents the minimum-norm solution of Eq. (2) and the second term is an arbitrary vector from the Jacobian null space.

The interest lies in a state trajectory that is optimal in some sense. The global optimization method proposes to use the system redundancy to derive an optimal trajectory. This can be achieved by means of methods of the calculus of variations.[14] To this end, we consider the augmented performance measure

$$L^* = \int_{t_0}^{t_f} \{\dot{q}^T\dot{q} + s^T[r - f(q)]\}\,\mathrm{d}t = \int_{t_0}^{t_f} \mu(q, \dot{q}, t)\,\mathrm{d}t \tag{6}$$

Then, the variation of L^* can be written as

$$\delta L^* = \int_{t_0}^{t_f} \left[\frac{\partial \mu}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mu}{\partial \dot{q}}\right)\right]^T \delta q(t)\,\mathrm{d}t + \left[\frac{\partial \mu}{\partial \dot{q}(t_f)}\right]^T \delta q(t_f) - \left[\frac{\partial \mu}{\partial \dot{q}(t_0)}\right]^T \delta q(t_0) = 0 \tag{7}$$

The coefficient of δq in the integrand yields the vector form of Euler's equation:

$$\frac{\partial \mu}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mu}{\partial \dot{q}}\right) = 0 \tag{8}$$

which leads to

$$J^T s - 2\ddot{q} = 0 \tag{9}$$

where J is the Jacobian as defined by Eq. (3). Differentiating Eq. (1) twice, we can write

$$J\ddot{q} = \ddot{r} - \dot{J}\dot{q} \tag{10}$$

so that, combining Eqs. (9) and (10) and augmenting the result with an obvious identity, we obtain the equations defining the optimal trajectory:

$$\dot{q} = \dot{q}, \qquad \ddot{q} = J^{\dagger}(q)(\ddot{r} - \dot{J}\dot{q}) \tag{11}$$

Note that, by letting $\dot{q} = v$, Eqs. (11) can be regarded as a set of $2n$ first-order state equations.

III. Dynamic Equations of Motion for Flexible Space Robot

We propose to derive the system equations of motion by means of Lagrange's equations in conjunction with a consistent kinematic scheme. To this end, we recall that body 0 is rigid, bodies 1 and 2 are flexible, and body 3 is rigid. Referring to Fig. 2, the displacement vector U and velocity vector V for a typical point in body 0 are as follows:

$$U_0 = R + C_0^T R_0 \tag{12a}$$

$$V_0 = \dot{R} + C_0^T \tilde{\omega}_0 R_0 \tag{12b}$$

Similarly, for body 1

$$U_1 = R + C_0^T L_0 + C_1^T(r_1 + u_1) \tag{13a}$$

$$V_1 = \dot{R} + C_0^T\tilde{\omega}_0 L_0 + C_1^T\tilde{\omega}_1(r_1 + u_1) + C_1^T\dot{u}_1 \tag{13b}$$

for body 2

$$U_2 = R + C_0^T L_0 + C_1^T(L_1 + u_{12}) + C_2^T(r_2 + u_2) \tag{14a}$$

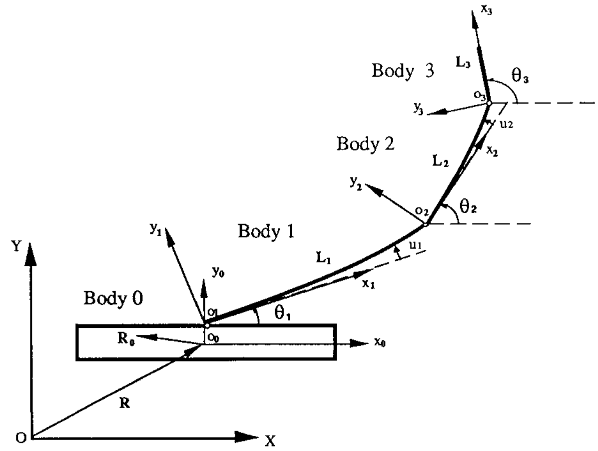

Fig. 2 Coordinate systems for space robot.

$$V_2 = \dot{R} + C_0^T\tilde{\omega}_0 L_0 + C_1^T\tilde{\omega}_1(L_1 + u_{12}) + C_1^T\dot{u}_{12} + C_2^T\tilde{\omega}_2(r_2 + u_2) + C_2^T\dot{u}_2 \tag{14b}$$

and for body 3

$$U_3 = R + C_0^T L_0 + C_1^T(L_1 + u_{12}) + C_2^T(L_2 + u_{23}) + C_3^T r_3 \tag{15a}$$

$$V_3 = \dot{R} + C_0^T\tilde{\omega}_0 L_0 + C_1^T\tilde{\omega}_1(L_1 + u_{12}) + C_1^T\dot{u}_{12} + C_2^T\tilde{\omega}_2(L_2 + u_{23}) + C_2^T\dot{u}_{23} + C_3^T\tilde{\omega}_3 r_3 \tag{15b}$$

where

$$C_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix} \quad i = 0, 1 \tag{16a}$$

$$C_2 = \begin{bmatrix} \cos(\theta_2 + \beta_1) & \sin(\theta_2 + \beta_1) \\ -\sin(\theta_2 + \beta_1) & \cos(\theta_2 + \beta_1) \end{bmatrix} \tag{16b}$$

$$C_3 = \begin{bmatrix} \cos(\theta_3 + \beta_1 + \beta_2) & \sin(\theta_3 + \beta_1 + \beta_2) \\ -\sin(\theta_3 + \beta_1 + \beta_2) & \cos(\theta_3 + \beta_1 + \beta_2) \end{bmatrix} \tag{16c}$$

$$\bar{M}_0 = f\begin{bmatrix} m_t & 0 & -S_{tx} & -S_{t1}s_1 - a_1 & -S_{t2}s_2 - a_2 & -S_3 s_3 \\ 0 & m_t & S_{ty} & S_{t1}c_1 - a_3 & S_{t2}c_2 - a_4 & S_3 c_3 \\ -S_{tx} & S_{ty} & I_{t0} & S_{t1}L_0 s_{10} - a_5 & S_{t2}L_0 s_{20} - a_6 & S_3 L_0 s_{30} \\ -S_{t1}s_1 - a_1 & S_{t1}c_1 - a_3 & S_{t1}L_0 s_{10} - a_5 & I_{t1} + z_1^T m_{77} z_1 & S_{t2}L_1 c_{21} + a_7 & S_3 L_1 c_{31} + a_8 \\ -S_{t2}s_2 - a_2 & S_{t2}c_2 - a_4 & S_{t2}L_0 s_{20} - a_6 & S_{t2}L_1 c_{21} + a_7 & I_{t2} + z_2^T m_{88} z_2 & S_3 L_2 c_{32} + a_9 \\ -S_3 s_3 & S_3 c_3 & S_3 L_0 s_{30} & S_3 L_1 c_{31} + a_8 & S_3 L_2 c_{32} + a_9 & I_3 \end{bmatrix} \tag{26}$$

are matrices of direction cosines, in which

$$\beta_1 = \left.\frac{\partial u_1}{\partial x_1}\right|_{x_1 = L_1}, \qquad \beta_2 = \left.\frac{\partial u_2}{\partial x_2}\right|_{x_2 = L_2} \tag{17}$$

are angular displacements due to flexibility,

$$\tilde{\omega}_0 = \begin{bmatrix} 0 & -\dot{\theta}_0 \\ \dot{\theta}_0 & 0 \end{bmatrix}, \qquad \tilde{\omega}_1 = \begin{bmatrix} 0 & -\dot{\theta}_1 \\ \dot{\theta}_1 & 0 \end{bmatrix}$$

$$\tilde{\omega}_2 = \begin{bmatrix} 0 & -(\dot{\theta}_2 + \dot{\beta}_1) \\ \dot{\theta}_2 + \dot{\beta}_1 & 0 \end{bmatrix} \tag{18}$$

$$\tilde{\omega}_3 = \begin{bmatrix} 0 & -(\dot{\theta}_3 + \dot{\beta}_1 + \dot{\beta}_2) \\ \dot{\theta}_3 + \dot{\beta}_1 + \dot{\beta}_2 & 0 \end{bmatrix}$$

are skew symmetric matrices representing the matrix counterpart of the $\boldsymbol{\omega}\times$ vector operation,

$$R = [x_0 \quad y_0]^T, \qquad r_1 = [x_1 \quad 0]^T, \qquad r_2 = [x_2 \quad 0]^T \tag{19}$$

are position vectors, and

$$u_1 = [0 \quad u_1]^T, \qquad u_2 = [0 \quad u_2]^T \tag{20}$$

are elastic displacement vectors. Moreover,

$$u_{12} = u_1|_{x_1 = L_1}, \qquad u_{23} = u_2|_{x_2 = L_2} \tag{21}$$

For convenience, we discretize the elastic displacements as follows:

$$u_i(x, t) = \Phi_i^T(x_i) z_i(t) \quad i = 1, 2 \tag{22}$$

where $\Phi_i(x)$ are vectors of quasicomparison functions[15] and $z_i(t)$ are vectors of generalized displacements. Regarding the robot arms as beams in bending, the quasicomparison functions can be chosen as a linear combination of the admissible functions:

$$\phi_k = \cosh\frac{\lambda_k x}{L} - \cos\frac{\lambda_k x}{L} - \sigma_k\left(\sinh\frac{\lambda_k x}{L} - \sin\frac{\lambda_k x}{L}\right) \quad k = 1, 2, \ldots \tag{23}$$

which represent the eigenfunctions of a clamped–free beam for k odd and clamped–clamped beam for k even.

Using Eqs. (12–23), the kinetic energy of the system can be written as

$$T = \sum_{i=0}^{3} T_i = \frac{1}{2}\sum_{i=0}^{3}\int_{\text{Body}\, i}\rho_i V_i^T V_i \, \mathrm{d}D_i = \tfrac{1}{2}\dot{q}^T M \dot{q} \tag{24}$$

where $q = [R^T \;\; \theta_0 \;\; \theta_1 \;\; \theta_2 \;\; \theta_3 \;\; z_1^T \;\; z_2^T]^T$ is the configuration vector and

$$M = \begin{bmatrix} & & & m_{17} & m_{18} \\ & \bar{M}_0 & & \vdots & \vdots \\ & & & m_{67} & m_{68} \\ m_{17}^T & \cdots & m_{67}^T & m_{77} & m_{78} \\ m_{18}^T & \cdots & m_{68}^T & m_{78}^T & m_{88} \end{bmatrix} \tag{25}$$

is the mass matrix, where

in which

$$a_1 = \bar{\Phi}_{t1}^T z_1 c_1, \qquad a_2 = \bar{\Phi}_{t2}^T z_2 c_2, \qquad a_3 = \bar{\Phi}_{t1}^T z_1 s_1$$
$$a_4 = \bar{\Phi}_{t2}^T z_2 s_2, \qquad a_5 = -\bar{\Phi}_{t1}^T z_1 L_0 c_{10} \tag{27}$$
$$a_6 = -\bar{\Phi}_{t2}^T z_2 L_0 c_{20}, \qquad a_7 = S_{t2}\Phi_{12}^T z_1 s_{21}$$
$$a_8 = S_3\Phi_{12}^T z_1 s_{31}, \qquad a_9 = S_3\Phi_{23}^T z_2 s_{32}$$

and

$$m_{17} = -\bar{\Phi}_{t1}^T s_1 - S_{t2}\Upsilon_{12}^T s_2 - S_3\Upsilon_{12}^T s_3$$
$$m_{27} = \bar{\Phi}_{t1}^T c_1 + S_{t2}\Upsilon_{12}^T c_2 + S_3\Upsilon_{12}^T c_3$$
$$m_{37} = \bar{\Phi}_{t1}^T L_0 s_{10} + S_{t2}L_0\Upsilon_{12}^T s_{20} + S_3 L_0\Upsilon_{12}^T s_{30}$$
$$m_{47} = \bar{\Phi}_1^T + (m_2 + m_3)L_1\Phi_{12}^T + S_{t2}L_1\Upsilon_{12}^T c_{21} + S_3 L_1\Upsilon_{12}^T c_{31}$$
$$m_{57} = S_{t2}\Phi_{12}^T c_{21} + I_{t2}\Upsilon_{12}^T + S_3 L_2\Upsilon_{12}^T c_{32}$$
$$m_{67} = S_3\Phi_{12}^T c_{31} + S_3 L_2\Upsilon_{12}^T c_{32} + I_3\Upsilon_{12}^T$$
$$m_{18} = -\bar{\Phi}_{t2}^T s_2 - S_3\Upsilon_{23}^T s_3$$
$$m_{28} = \bar{\Phi}_{t2}^T c_2 + S_3\Upsilon_{23}^T c_3$$
$$m_{38} = \bar{\Phi}_{t2}^T L_0 s_{20} + S_3 L_0\Upsilon_{23}^T s_{30}$$
$$m_{48} = \bar{\Phi}_{t2}^T L_1 c_{21} + S_3 L_1\Upsilon_{23}^T c_{31} \tag{28}$$
$$m_{58} = \bar{\Phi}_2^T + m_3 L_2\Phi_{23}^T + S_3 L_2\Upsilon_{23}^T c_{32}$$
$$m_{68} = S_3\Phi_{23}^T c_{32} + I_3\Upsilon_{23}^T$$

$$m_{77} = \Lambda_1 + (m_2 + m_3)\Phi_{12}\Phi_{12}^T + (I_{t2} + I_3)\Upsilon_{12}\Upsilon_{12}^T + S_{t2}\left(\Phi_{12}\Upsilon_{12}^T + \Upsilon_{12}\Phi_{12}^T\right)c_{21} + S_3\left(\Phi_{12}\Upsilon_{12}^T + \Upsilon_{12}\Phi_{12}^T\right)c_{31} + 2S_3 L_2\Upsilon_{12}\Upsilon_{12}^T c_{32}$$

$$m_{78} = \tilde{\mathbf{\Phi}}_2 \mathbf{\Upsilon}_{12}^T + m_3 L_2 \mathbf{\Phi}_{23} \mathbf{\Upsilon}_{12}^T + \bar{\mathbf{\Phi}}_{t2} \mathbf{\Phi}_{12}^T c_{21} + S_3 \mathbf{\Upsilon}_{23} \left(\mathbf{\Phi}_{12}^T + L_2 \mathbf{\Upsilon}_{12}^T\right) c_{32} + I_3 \mathbf{\Upsilon}_{23} \mathbf{\Upsilon}_{12}^T + S_3 \mathbf{\Phi}_{23} \mathbf{\Upsilon}_{12}^T c_{32}$$

$$m_{88} = \Lambda_2 + m_3 \mathbf{\Phi}_{23} \mathbf{\Phi}_{23}^T + I_3 \mathbf{\Upsilon}_{23} \mathbf{\Upsilon}_{23}^T + S_3 \left(\mathbf{\Phi}_{23} \mathbf{\Upsilon}_{23}^T + \mathbf{\Upsilon}_{23} \mathbf{\Phi}_{23}^T\right) c_{32}$$

and we note that $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, $s_{ij} = \sin(\theta_i - \theta_j)$, and $c_{ij} = \cos(\theta_i - \theta_j)$. Moreover, we have used the following definitions:

$$m_t = m_0 + m_1 + m_2 + m_3$$

$$S_{tx} = S_{0x} \sin\theta_0 + S_{0y} \cos\theta_0 + (m_1 + m_2 + m_3) L_0 \cos\theta_0$$

$$S_{ty} = S_{0x} \cos\theta_0 - S_{0y} \sin\theta_0 - (m_1 + m_2 + m_3) L_0 \sin\theta_0$$

$$S_{t1} = S_1 + (m_2 + m_3) L_1, \qquad S_{t2} = S_2 + m_3 L_2 \tag{29}$$

$$I_{t0} = I_{0x} + I_{0y} + (m_1 + m_2 + m_3) L_0^2$$

$$I_{t1} = I_1 + (m_2 + m_3) L_1^2, \qquad I_{t2} = I_2 + m_3 L_2^2$$

$$\bar{\mathbf{\Phi}}_{t1} = \bar{\mathbf{\Phi}}_1 + (m_2 + m_3) \mathbf{\Phi}_{12}, \qquad \bar{\mathbf{\Phi}}_{t2} = \bar{\mathbf{\Phi}}_2 + m_3 \mathbf{\Phi}_{23}$$

in which

$$m_i = \int_{\text{Body } i} \rho_i \, \mathrm{d}D_i \qquad i = 0, 1, 2, 3$$

$$S_i = \int_{\text{Body } i} \rho_i x_i \, \mathrm{d}D_i, \qquad I_i = \int_{\text{Body } i} \rho_i x_i^2 \, \mathrm{d}D_i \qquad i = 1, 2, 3$$

$$S_{0x} = \int_{\text{Body } 0} \rho_0 x \, \mathrm{d}D_0, \qquad S_{0y} = \int_{\text{Body } 0} \rho_0 y \, \mathrm{d}D_0$$

$$I_{0x} = \int_{\text{Body } 0} \rho_0 x^2 \, \mathrm{d}D_0, \qquad I_{0y} = \int_{\text{Body } 0} \rho_0 y^2 \, \mathrm{d}D_0 \tag{30}$$

$$\bar{\mathbf{\Phi}}_i = \int_{\text{Body } i} \rho_i \mathbf{\Phi}_i \, \mathrm{d}D_i, \qquad \tilde{\mathbf{\Phi}}_i = \int_{\text{Body } i} \rho_i x_i \mathbf{\Phi}_i \, \mathrm{d}D_i$$

$$\Lambda_i = \int_{\text{Body } i} \rho_i \mathbf{\Phi}_i \mathbf{\Phi}_i^T \, \mathrm{d}D_i \qquad i = 1, 2$$

$$\mathbf{\Phi}_{12} = \mathbf{\Phi}_1(x_1)\big|_{x_1 = L_1}, \qquad \mathbf{\Phi}_{23} = \mathbf{\Phi}_2(x_2)\big|_{x_2 = L_2}$$

$$\mathbf{\Upsilon}_{12} = \mathbf{\Phi}_1'(x_1)\big|_{x_1 = L_1}, \qquad \mathbf{\Upsilon}_{23} = \mathbf{\Phi}_2'(x_2)\big|_{x_2 = L_2}$$

The potential energy for the system is due entirely to the elasticity of the robot arms and can be written in the form

$$V = \sum_{i=1}^{2} \tfrac{1}{2} z_i^T K_i z_i = \tfrac{1}{2} \boldsymbol{q}^T K \boldsymbol{q} \tag{31}$$

where

$$K = \text{block-diag}[\,0 \quad K_1 \quad K_2\,] \tag{32}$$

in which

$$K_i = \int_0^{L_i} E I_i \mathbf{\Phi}_i''(\mathbf{\Phi}_i'')^T \, \mathrm{d}x_i \quad i = 1, 2 \tag{33}$$

are the stiffness matrices of bodies i; EI_i denotes the bending stiffnesses. Note that the gravitational potential is ignored here on the basis that it is negligibly small.

The virtual work of the system is

$$\delta W = F_{x0}\,\delta x_0 + F_{y0}\,\delta y_0 + \tau_0\,\delta\theta_0 + \tau_1\,\delta\theta_1 + \tau_2(\delta\theta_2 + \delta\beta_1) + \tau_3(\delta\theta_3 + \delta\beta_1 + \delta\beta_2) + C_1^T \sum_{i=1}^{m_1} \boldsymbol{f}_{1i}^T\,\delta \boldsymbol{U}_1(x_{1i}) + C_2^T \sum_{i=1}^{m_2} \boldsymbol{f}_{2i}^T\,\delta \boldsymbol{U}_2(x_{2i}) = \boldsymbol{Q}^T\,\delta\boldsymbol{q} \tag{34}$$

where $\boldsymbol{Q}$ is a generalized force vector. As shown in Fig. 1, τ_j are control torques acting on bodies j at $x_j = 0^+$ $(j = 1, 2)$ and $\boldsymbol{f}_{ji}$ $(i = 1, 2, \ldots, m_j)$ are control forces acting throughout bodies j $(j = 1, 2)$. Lagrange's equations for the system can be expressed in the symbolic vector form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial T}{\partial \boldsymbol{q}} + \frac{\partial V}{\partial \boldsymbol{q}} = \boldsymbol{Q} \tag{35}$$

Inserting Eqs. (24), (31), and (34) into Eq. (35), we obtain

$$M\ddot{\boldsymbol{q}} + \dot{M}\dot{\boldsymbol{q}} - \tfrac{1}{2}\dot{\boldsymbol{q}}^T \frac{\partial M}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} + K\boldsymbol{q} = \boldsymbol{Q} \tag{36}$$

where we recognize that $M = M(\boldsymbol{q})$ and that $\dot{\boldsymbol{q}}^T(\partial M/\partial \boldsymbol{q})\dot{\boldsymbol{q}}$ is a vector with entries $\dot{\boldsymbol{q}}^T(\partial M/\partial q_j)\dot{\boldsymbol{q}}$ $(j = 1, 2, \ldots, n)$.

IV. Perturbation Method

The approach described above involves trajectory optimization and determination of the forces and torques required for trajectory realization from the dynamic equations of motion, a procedure referred to as inverse dynamics. This approach has been used widely in robotics, but application to date have been confined to rigid robots. Indeed, flexibility is likely to cause serious difficulties not only in trajectory optimization but also in inverse dynamics. In most cases of interest, however, elastic motions tend to be small compared to maneuvering motions, particularly when the elastic vibration is controlled. In such cases, it is natural to use a perturbation method to separate the problem into two problems involving quantities of different orders of magnitude. One problem, referred to as a zero-order problem, is concerned with maneuvering of the robot as if it were rigid. The second problem, referred to as a first-order problem, is concerned with the control of the elastic vibrations and perturbations from the rigid-body maneuvering. Of course, the assumption is that the variables in the zero-order problem are one order of magnitude larger than the variables in the first-order problem.

In view of the above, we express the configuration vector and associated generalized force vector in the form

$$\boldsymbol{q}(t) = \boldsymbol{q}_0(t) + \boldsymbol{q}_1(t), \qquad \boldsymbol{Q}(t) = \boldsymbol{Q}_0(t) + \boldsymbol{Q}_1(t) \tag{37}$$

where the subscripts 0 and 1 denote the different orders of magnitude. Introducing Eqs. (37) into Eq. (36) and separating quantities of different orders of magnitude, we obtain the equation defining the zero-order problem

$$M_0(\boldsymbol{q}_0)\ddot{\boldsymbol{q}}_0 + B_0(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)\dot{\boldsymbol{q}}_0 = \boldsymbol{Q}_0 \tag{38}$$

where B_0 is a coefficient matrix, and the equation defining the first-order problem

$$M_1(\boldsymbol{q}_0)\ddot{\boldsymbol{q}}_1 + G(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)\dot{\boldsymbol{q}}_1 + C(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0, \ddot{\boldsymbol{q}}_0)\boldsymbol{q}_1 = \boldsymbol{d}(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0, \ddot{\boldsymbol{q}}_0) + \boldsymbol{Q}_1 \tag{39}$$

where G and C are coefficient matrices and $\boldsymbol{d}$ is a persistent disturbing vector. Explicit expressions for M_0, B_0, M_1, G, C, and $\boldsymbol{d}$ are given in the Appendix.

It is clear from the above that the zero-order problem can be solved independently of the first-order problem, which is typical of perturbation solutions. On the other hand, the first-order problem depends on the solution to the zero-order problem, where the dependence manifests itself in the form of time-varying coefficients and persistent disturbances. The zero-order problem is nonlinear and of relatively low order. In contrast, the first-order problem is linear, albeit time varying, and of relatively large order. The zero-order equation (38) can be used for trajectory planning and inverse dynamics in conjunction with the rigid-body maneuvering. The first-order equation (39) can be used to design feedback controls for the elastic vibrations and the perturbations from the rigid-body maneuvering.

V. Rigid-Body Maneuvering

As indicated in Sec. IV, we propose to carry out trajectory planning on the basis of the robot system regarded as an articulated, rigid-multibody system. Then, the force vector $\boldsymbol{Q}_0$ permitting realization of the rigid-body maneuver is obtained from Eq. (38).

Assuming that the vector $\boldsymbol{r}(t)$ representing the motion of the end point of the end effector/payload is given, we can obtain the rigid-robot trajectory $\boldsymbol{q}_0 = [x_0 \;\; y_0 \;\; \theta_{00} \;\; \theta_{01} \;\; \theta_{02} \;\; \theta_{03} \;\; \boldsymbol{0}^T \;\; \boldsymbol{0}^T]^T$ by solving Eqs. (11). We rewrite Eqs. (11) in the form

$$\dot{\boldsymbol{q}} = \boldsymbol{v}, \qquad \dot{\boldsymbol{v}} = J^{\dagger}(\boldsymbol{q})(\ddot{\boldsymbol{r}} - \dot{J}\dot{\boldsymbol{q}}) \tag{40}$$

which are subject to the boundary conditions

$$\boldsymbol{q}(t_0) = \boldsymbol{c}, \qquad \boldsymbol{v}(t_f) = J^{\dagger}(\boldsymbol{q}(t_f), t_f)\dot{\boldsymbol{r}}(t_f) \tag{41}$$

where t_0 was taken as zero and $\boldsymbol{c}$ is a constant vector.

Equations (40) and (41) constitute a nonlinear two-point boundary value problem. The solution of such problems is generally very difficult to obtain, particularly for systems of relatively large orders. The shooting method reduces the problem to the solution of the nonlinear equation

$$F(w) = 0 \tag{42}$$

which can be solved by Newton's method.[16] The approach suffers from sensitivity to initial guesses, however, and can lead to divergence. Improving the initial guess is not always feasible, particularly for multivariable systems. As an alternative, one can consider the continuation method. According to this method, Eq. (41) is replaced by a family of problems given by

$$\boldsymbol{\Gamma}(\alpha, \boldsymbol{w}) = \alpha \boldsymbol{F}(\boldsymbol{w}) + (1 - \alpha)\boldsymbol{S}(\boldsymbol{w}) = \boldsymbol{0} \tag{43}$$

where $\alpha \in [0,\ 1]$ is a parameter and $\boldsymbol{S}(\boldsymbol{w})$ is a function such that the equation $\boldsymbol{S}(\boldsymbol{w}) = \boldsymbol{0}$ is relatively easy to solve. This is the equation corresponding to $\alpha = 0$. At $\alpha = 1$, Eq. (43) reduces to the equation we would like to solve. The approach consists of solving Eq. (43) in a step-by-step manner, beginning with $\alpha = 0$ and finishing with $\alpha = 1$. The solution at every step uses as an initial guess the solution obtained in the previous step. If Newton's method is used to solve Eq. (43), then failure can occur, because Newton's method postulates a monotonic increase in α, and in a convergent solution α does not necessarily increase monotonically.

Quite recently, a new version of the continuation method, known as probability-1 homotopy algorithms,[13] has been developed. The algorithms are relatively insensitive to initial guesses; i.e., they are capable of converging to the correct solution even for initial guesses not very close to the solution. Based on the homotopy theory, α is allowed to increase or decrease arbitrarily within $s \in [0,\ s^*]$ as long as the curve $\boldsymbol{\Gamma} = \boldsymbol{0}$ is followed, which is a distinct advantage over Newton's continuation method, for which α is required to increase monotonically.

For the trajectory planning at hand, we use the homotopy method to solve the nonlinear two-point boundary value problem defined by Eqs. (40) and (41). The homotopy map is given by Eq. (43) in which

$$\boldsymbol{F}(\boldsymbol{w}) = \dot{\boldsymbol{q}}(t_f, \boldsymbol{w}) - J^{\dagger}(t_f, \boldsymbol{w})\dot{\boldsymbol{r}}(t_f) \tag{44a}$$

$$\boldsymbol{S}(\boldsymbol{w}) = \boldsymbol{w} - \boldsymbol{a} \tag{44b}$$

where $\boldsymbol{w} = \dot{\boldsymbol{q}}(0)$ and $\boldsymbol{a}$ is an initial guess of $\boldsymbol{w}$.

VI. Control of Elastic Vibrations and Perturbations from Rigid-Body Maneuver

At this point, we turn our attention to the design of controls for the elastic vibrations and perturbations from the rigid-body maneuver. To this end, we adjoin the identity $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}$ and rewrite Eq. (39) in the state form

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t) + \bar{\boldsymbol{d}}(t) \tag{45}$$

where

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q}_1 \\ \dot{\boldsymbol{q}}_1 \end{bmatrix}, \quad B_f \boldsymbol{u}(t) = \boldsymbol{Q}_1, \quad \bar{\boldsymbol{d}} = \begin{bmatrix} 0 \\ M^{-1}\boldsymbol{d} \end{bmatrix} \tag{46}$$

are the state vector, control vector, and disturbance vector, respectively, and

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}C & -M^{-1}G \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1}B_f \end{bmatrix} \tag{47}$$

are coefficient matrices, in which

$$B_f = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -\sin\theta_1 & \cdots & -\sin\theta_1 & -\sin\theta_2 & \cdots & -\sin\theta_2 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\cos\theta_1 & \cdots & -\cos\theta_1 & -\cos\theta_2 & \cdots & -\cos\theta_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & b_{01} & \cdots & b_{01} & b_{02} & \cdots & b_{02} \\ 0 & 0 & 0 & 1 & 0 & 0 & x_{11} & \cdots & x_{1m} & b_1 & \cdots & b_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & x_{21} & \cdots & x_{2m} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Upsilon}_{12} & \boldsymbol{\Upsilon}_{12} & \boldsymbol{\Phi}_1(x_{11}) & \cdots & \boldsymbol{\Phi}_1(x_{1m}) & \boldsymbol{b}_{21} & \cdots & \boldsymbol{b}_{2m} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Upsilon}_{23} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{\Phi}_2(x_{21}) & \cdots & \boldsymbol{\Phi}_2(x_{2m}) \end{bmatrix} \tag{48}$$

where

$$\boldsymbol{b}_{2i} = x_{2i}\boldsymbol{\Upsilon}_{12} + \boldsymbol{\Phi}_{12}\cos(\theta_2 - \theta_1)$$
$$b_{01} = L_0 \sin(\theta_1 - \theta_0) \qquad b_{02} = L_0\cos(\theta_2 - \theta_0) \tag{49}$$
$$b_1 = L_1\cos(\theta_2 - \theta_1)$$

We propose to compensate for the persistent disturbance open-loop control and to control the elastic vibrations and rigid-body perturbations in the absence of persistent disturbance closed-loop control. Hence, we divide the control vector into

$$\boldsymbol{u} = \boldsymbol{u}_o + \boldsymbol{u}_c \tag{50}$$

where the open-loop control has the form

$$\boldsymbol{u}_o(t) = -B_f^{\dagger}(t)\boldsymbol{d}(t) \tag{51}$$

so that Eq. (45) reduces to

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}_c(t) + D(t)\boldsymbol{d}(t) \tag{52}$$

$$D(t) = \begin{bmatrix} 0 \\ M^{-1}[I - B_f(t)B_f^{\dagger}(t)] \end{bmatrix} \tag{53}$$

We wish to determine the feedback control in an optimal fasion. To this end, we use the LQR theory for which the performance measure has the form

$$L = \boldsymbol{x}^T(t_f)H_f\boldsymbol{x}(t_f) + \int_{t_0}^{t_f} \left[\boldsymbol{x}^T(t)Q(t)\boldsymbol{x}(t) + \boldsymbol{u}_c^T(t)R(t)\boldsymbol{u}_c(t)\right] \mathrm{d}t \tag{54}$$

It is well known that the optimal control law is given by[17]

$$\boldsymbol{u}_c(t) = -R^{-1}(t)B^T(t)K(t)\boldsymbol{x}(t) \tag{55}$$

where $K(t)$ satisfies the matrix differential Riccati equation

$$\dot{K}(t) = -K(t)A(t) - A^T(t)K(t) - Q(t) + K(t)B(t)R^{-1}(t)B^T(t)K(t) \qquad K(t_f) = H_f \tag{56}$$

VII. Numerical Example

The numerical example is concerned with the flexible robot shown in Fig. 1. The solution follows the pattern established earlier, i.e., trajectory planning based on the rigid robot and feedback control based on the first-order perturbations.

A. Trajectory Planning

As the manipulation variable vector, we use $\boldsymbol{r} = [x \ y \ \varphi]^T$ where x and y are the cartesian components of the payload tip and φ is a measure of the orientation of the end effector/payload. For the sake of this example, we assume that the platform undergoes translation only ($\theta_0 = 0$), so that the configuration vector $\boldsymbol{q} = [x_0 \ y_0 \ \theta_1 \ \theta_2 \ \theta_3]^T$, where for simplicity we omitted the subscript 0 identifying zero-order quantities. Hence, the relation between $\boldsymbol{r}$ and $\boldsymbol{q}$ is by components

$$x = x_0 + L_1\cos\theta_1 + L_2\cos\theta_2 + L_3\cos\theta_3$$
$$y = y_0 + L_0 + L_1\sin\theta_1 + L_2\sin\theta_2 + L_3\sin\theta_3 \tag{57}$$
$$\varphi = \cos\theta_3$$

For the purpose of this example, we choose

$$x = 7.5\cos 0.2\pi t + 7.5$$
$$y = -7.5\cos 0.2\pi t + 7.5 \qquad t \in [0, 5.0] \tag{58}$$
$$\varphi = \tfrac{1}{2}\sqrt{2} - 0.2\sqrt{2t}$$

so that at $t = 0$ the tip starts from the position [15 m 0] and orientation of 45 deg and at $t = 5$ s it ends in the position [0 15 m] and orientation of 135 deg. The tip velocity is zero at $t = 0$ and $t = 5$ s.

From Eqs. (57), the Jacobian matrix is

$$J = \begin{bmatrix} 1 & 0 & -L_1\sin\theta_1 & -L_2\sin\theta_2 & -L_3\sin\theta_3 \\ 0 & 1 & L_1\cos\theta_1 & L_2\cos\theta_2 & L_3\cos\theta_3 \\ 0 & 0 & 0 & 0 & -\sin\theta_3 \end{bmatrix} \tag{59}$$

and the performance measure for the optimal trajectory is as given by Eqs. (6) and (10).

Letting $L_0 = 2.5$ m, $L_1 = L_2 = 10.0$ m, and $L_3 = 2.0$ m, the trajectory is computed for the two cases:

Case 1:

$$x_0 = -6.4142 \text{ m}, \quad y_0 = -3.9142 \text{ m}, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = \pi/4$$

Case 2:

$$x_0 = -3.4853 \text{ m}, \quad y_0 = -10.9853 \text{ m}, \quad \theta_1 = 0, \quad \theta_2 = \theta_3 = \pi/4$$

Figure 3 shows time-lapse pictures of the robot configuration for cases 1 and 2. The results were obtained by the homotopy method. Note that, although the initial configuration is different, in both cases the initial tip position is [15 m 0] and the initial end effector orientation is 45 deg, as can be seen from Fig. 3.

Using inverse dynamics, the forces and torques required for rigid-body maneuvering were computed by means of Eq. (38).

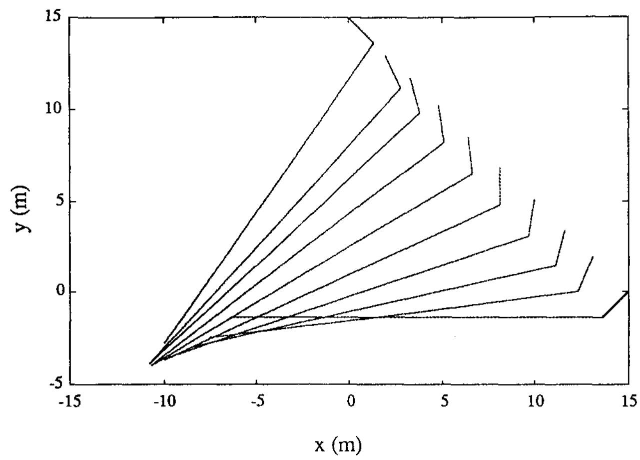

Fig. 3a Time-lapse picture of robot for case 1.

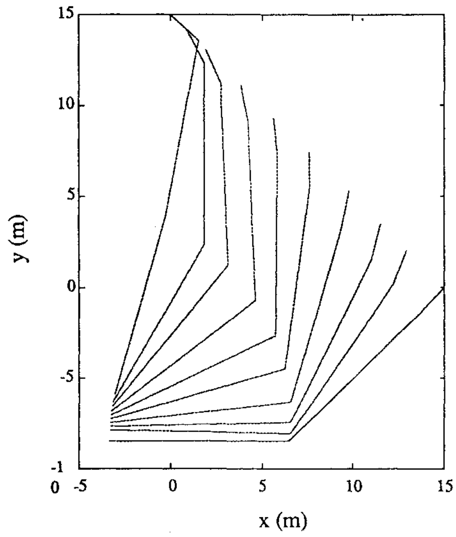

Fig. 3b Time-lapse picture of robot for case 2.

B. Optimal Feedback Control

The values of the system parameters used are as follows:

$$m_0 = 40\,\text{kg}, \qquad m_1 = m_2 = 10\,\text{kg}, \qquad m_3 = 2\,\text{kg}$$
$$L_0 = 2.5\,\text{m}, \qquad L_1 = L_2 = 10\,\text{m}, \qquad L_3 = 2\,\text{m}$$
$$S_x = S_y = 0$$
$$I_x = 83.333\,\text{kg-m}^2, \qquad I_y = 333.333\,\text{kg-m}^2$$
$$EI_1 = EI_2 = 10^4\,\text{kg-m}^2$$

The elastic displacement for the two arms was modeled by means of five quasicomparison functions. The coefficient matrices for the performance measure [Eq. (54)], were chosen as

$$Q = \text{diag}(10^3 \ \cdots \ 10^3 \ 0 \ \cdots \ 0)$$
$$R = \text{diag}(1.0 \ \cdots \ 1.0), \qquad H_f = 0$$

Numerical solutions of the Riccati equation were obtained by an algorithm described in Ref. 18. Time histories of the uncontrolled and controlled responses are shown in Fig. 4.

VIII. Summary and Conclusions

A flexible space robot can be regarded as an articulated flexible multibody system. Typical missions of space robots are the

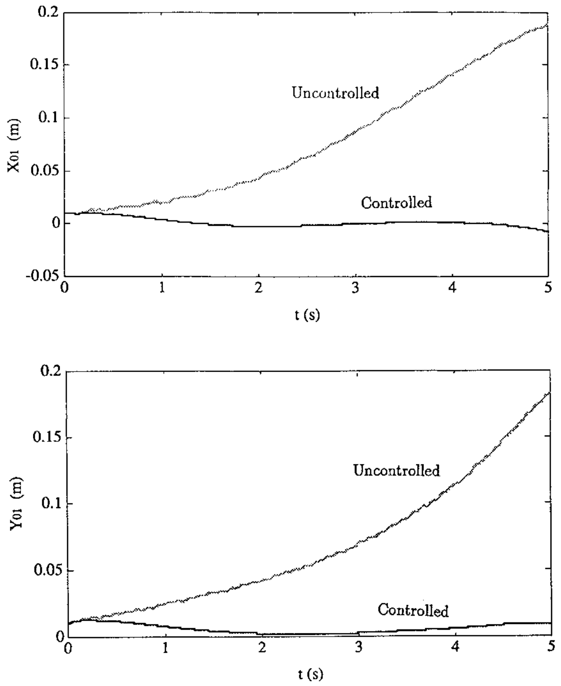

Fig. 4a Time history of controlled and uncontrolled perturbations in rigid-body translations.

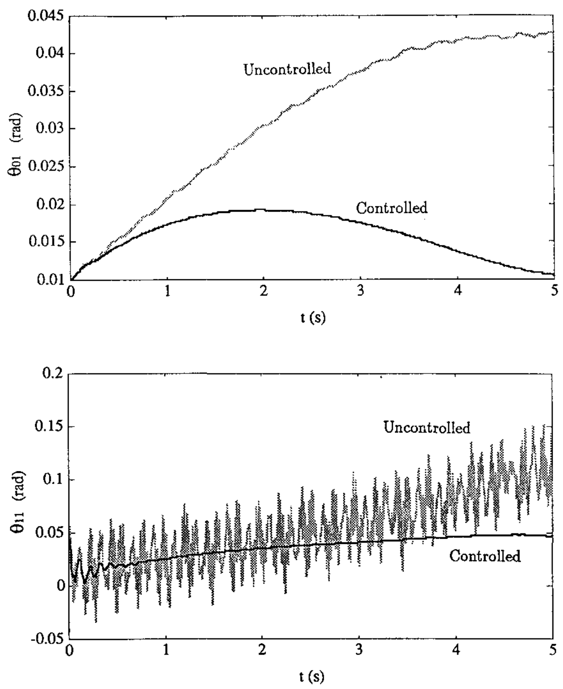

Fig. 4b Time history of controlled and uncontrolled perturbations in rigid-body rotations of bodies 0 and 1.

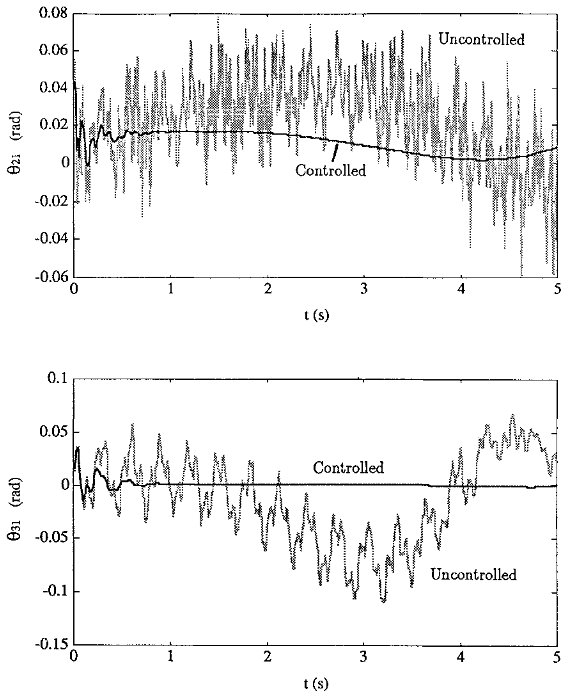

Fig. 4c Time history of controlled and uncontrolled perturbations in rigid-body rotations of bodies 2 and 3.

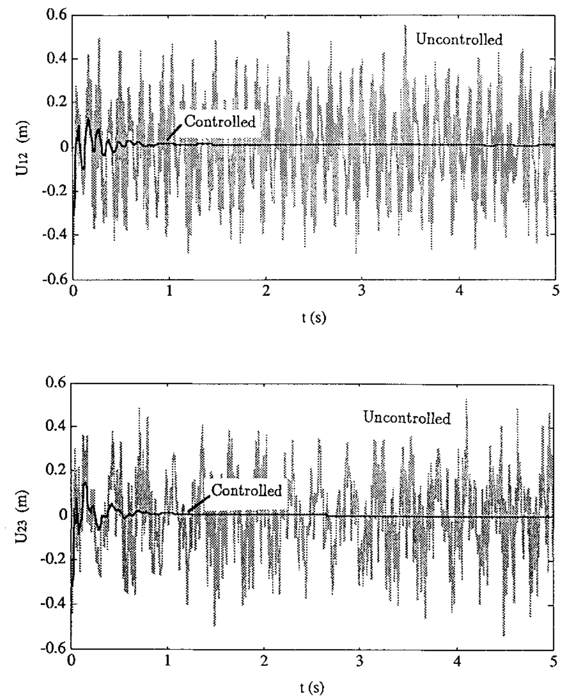

Fig. 4d Time history of tip elastic displacement of bodies 1 and 2.

collection of space debris, recovery of spacecraft stranded in a useless orbit, repair of orbiting spacecraft, construction of a space station in orbit, and servicing the space station while in operation. In all cases, the task of the robot is to maneuver payloads.

This paper is concerned with a space robot consisting of a rigid platform, such as the Space Shuttle, two articulated flexible arms, and a rigid end effector/payload. The task is to ferry some payload and to dock smoothly with an orbiting target. Under the assumption that maneuver motions are much larger than elastic motions, a perturbation approach permits dividing the problem into a zero-order problem (in a perturbation sense) for the rigid-body maneuvering of the robot and a first-order problem for the control of the elastic motions, as well as of perturbations from the rigid-body maneuvering motions. The zero-order control is open loop and the first-order control is closed loop.

The robot under consideration possesses redundancies, because the payload maneuvering is defined by three variables and the rigid-body robot maneuvering is defined by six variables. Assuming a certain payload maneuvering trajectory, the trajectory of the rigid robot is optimized by means of the global optimization method. Then, using inverse dynamics, the control forces and torques required for the rigid-body maneuvering are determined. The feedback control forces and torques required for the control of the elastic vibration and perturbations from the rigid-body motions are determined by the LQR theory. A numerical example demonstrates the approach.

Appendix

The mass matrix M_0 appearing in Eq. (38) is defined as

$$\boldsymbol{M}_0 = \begin{bmatrix} m_t & 0 & -S_{tx} & -S_{t1}s_1 & -S_{t2}s_2 & -S_3s_3 \\ 0 & m_t & -S_{ty} & S_{t1}c_1 & S_{t2}c_2 & S_3c_3 \\ -S_{tx} & -S_{ty} & I_{t0} & S_{t1}L_0s_{10} & S_{t2}L_0s_{20} & S_3L_0s_{30} \\ -S_{t1}s_1 & S_{t1}c_1 & S_{t1}L_0s_{10} & I_{t1} & S_{t2}L_1c_{21} & S_3L_1c_{31} \\ -S_{t2}s_2 & S_{t2}c_2 & S_{t2}L_0s_{20} & S_{t2}L_1c_{21} & I_{t2} & S_3L_2c_{32} \\ -S_3s_3 & S_3c_3 & S_3L_0s_{30} & S_3L_1c_{31} & S_3L_2c_{32} & I_3 \end{bmatrix} \tag{A1}$$

and the coefficient matrix B_0 is given by

$$B_0 = \begin{bmatrix} 0 & 0 & -S_{ty}\dot\theta_0 & -S_{t1}c_1\dot\theta_1 & -S_{t2}c_2\dot\theta_2 & -S_3c_3\dot\theta_3 \\ 0 & 0 & -S_{tx}\dot\theta_0 & -S_{t1}s_1\dot\theta_1 & -S_{t2}s_2\dot\theta_2 & -S_3s_3\dot\theta_3 \\ 0 & 0 & 0 & S_{t1}L_0c_{10}\dot\theta_1 & S_{t2}L_0c_{20}\dot\theta_2 & S_3L_0c_{30}\dot\theta_3 \\ 0 & 0 & -S_{t1}L_0c_{10}\dot\theta_0 & 0 & -S_{t2}L_1s_{21}\dot\theta_2 & -S_3L_1s_{31}\dot\theta_3 \\ 0 & 0 & -S_{t2}L_0c_{20}\dot\theta_0 & S_{t2}L_1s_{21}\dot\theta_1 & 0 & -S_3L_2s_{32}\dot\theta_3 \\ 0 & 0 & -S_3L_0c_{30}\dot\theta_0 & S_3L_1s_{31}\dot\theta_1 & S_3L_2s_{32}\dot\theta_2 & 0 \end{bmatrix} \tag{A2}$$

Moreover, the mass matrix in Eq. (39) has the form

$$M_1 = \begin{bmatrix} & & & \boldsymbol{m}_{17} & \boldsymbol{m}_{18} \\ & M_0 & & \vdots & \vdots \\ & & & \boldsymbol{m}_{67} & \boldsymbol{m}_{68} \\ \boldsymbol{m}_{17}^T & \cdots & \boldsymbol{m}_{67}^T & m_{77} & m_{78} \\ \boldsymbol{m}_{18}^T & \cdots & \boldsymbol{m}_{68}^T & m_{78}^T & m_{88} \end{bmatrix} \tag{A3}$$

and the coefficient matrices G and C are given by

$$G = \begin{bmatrix} & & & & & & \boldsymbol{G}_{17} & \boldsymbol{G}_{18} \\ & & & & & & \boldsymbol{G}_{27} & \boldsymbol{G}_{28} \\ & & & & & & \boldsymbol{G}_{37} & \boldsymbol{G}_{38} \\ & & & 2B_0 & & & \boldsymbol{G}_{47} & \boldsymbol{G}_{48} \\ & & & & & & \boldsymbol{G}_{57} & \boldsymbol{G}_{58} \\ & & & & & & \boldsymbol{G}_{67} & \boldsymbol{G}_{68} \\ 0 & 0 & \boldsymbol{G}_{73} & \boldsymbol{G}_{74} & \boldsymbol{G}_{75} & \boldsymbol{G}_{76} & G_{77} & G_{78} \\ 0 & 0 & \boldsymbol{G}_{83} & \boldsymbol{G}_{84} & \boldsymbol{G}_{85} & \boldsymbol{G}_{86} & G_{87} & G_{88} \end{bmatrix} \tag{A4}$$

where

$$\boldsymbol{G}_{17} = -2\bar{\boldsymbol{\Phi}}_{t1}^T c_1\dot\theta_1 - 2S_{t2}\boldsymbol{\Upsilon}_{12}^T c_2\dot\theta_2 - 2S_3\boldsymbol{\Upsilon}_{12}^T c_3\dot\theta_3$$

$$\boldsymbol{G}_{18} = -2\bar{\boldsymbol{\Phi}}_{t2}^T c_2\dot\theta_2 - 2S_3\boldsymbol{\Upsilon}_{23}^T c_3\dot\theta_3$$

$$\boldsymbol{G}_{27} = -2\bar{\boldsymbol{\Phi}}_{t1}^T s_1\dot\theta_1 - 2S_{t2}\boldsymbol{\Upsilon}_{12}^T s_2\dot\theta_2 - 2S_3\boldsymbol{\Upsilon}_{12}^T s_3\dot\theta_3$$

$$\boldsymbol{G}_{28} = -2\bar{\boldsymbol{\Phi}}_{t2}^T s_2\dot\theta_2 - 2S_3\boldsymbol{\Upsilon}_{23}^T s_3\dot\theta_3$$

$$\boldsymbol{G}_{37} = 2\bar{\boldsymbol{\Phi}}_{t1}^T L_0c_{10}\dot\theta_1 + 2S_{t2}L_0\boldsymbol{\Upsilon}_{12}^T c_{20}\dot\theta_2 + 2S_3L_0\boldsymbol{\Upsilon}_{12}^T c_{30}\dot\theta_3$$

$$\boldsymbol{G}_{38} = 2\bar{\boldsymbol{\Phi}}_{t2}^T L_0c_{20}\dot\theta_2 + 2S_3L_0\boldsymbol{\Upsilon}_{23}^T c_{30}\dot\theta_3$$

$$\boldsymbol{G}_{47} = -2S_{t2}L_1\boldsymbol{\Upsilon}_{12}^T s_{21}\dot\theta_2 - 2S_3L_1\boldsymbol{\Upsilon}_{12}^T s_{31}\dot\theta_3$$

$$\boldsymbol{G}_{48} = -2\bar{\boldsymbol{\Phi}}_{t2}^T L_1s_{21}\dot\theta_2 - 2S_3L_1\boldsymbol{\Upsilon}_{23}^T s_{31}\dot\theta_3$$

$$\boldsymbol{G}_{57} = 2S_{t2}\boldsymbol{\Phi}_{12}^T s_{21}\dot\theta_1 - 2S_3L_2\boldsymbol{\Upsilon}_{12}^T s_{32}\dot\theta_3$$

$$\boldsymbol{G}_{58} = -2S_3L_2\boldsymbol{\Upsilon}_{23}^T s_{32}\dot\theta_3$$

$$\boldsymbol{G}_{67} = 2S_3\boldsymbol{\Phi}_{12}^T s_{31}\dot\theta_1 + 2S_3L_2\boldsymbol{\Upsilon}_{12}^T s_{32}\dot\theta_2$$

$$\boldsymbol{G}_{68} = 2S_3\boldsymbol{\Phi}_{23}^T s_{32}\dot\theta_2$$

$$\boldsymbol{G}_{73} = -2\bar{\boldsymbol{\Phi}}_{t1}L_0c_{10}\dot\theta_0 - 2S_{t2}\boldsymbol{\Upsilon}_{12}L_0c_{20}\dot\theta_0 - 2S_3L_0\boldsymbol{\Upsilon}_{12}c_{30}\dot\theta_0$$

$$\boldsymbol{G}_{74} = 2S_{t2}\boldsymbol{\Upsilon}_{12}s_{21}\dot\theta_1 + 2S_3L_1\boldsymbol{\Upsilon}_{12}s_{31}\dot\theta_1$$

$$\boldsymbol{G}_{75} = -2S_{t2}\boldsymbol{\Phi}_{12}s_{21}\dot\theta_2 + 2S_3L_2\boldsymbol{\Upsilon}_{12}s_{32}\dot\theta_2$$

$$\boldsymbol{G}_{76} = -2S_3\boldsymbol{\Phi}_{12}s_{31}\dot\theta_3 - 2S_3L_2\boldsymbol{\Upsilon}_{12}s_{32}\dot\theta_3 \tag{A5}$$

$$G_{77} = \left(\boldsymbol{\Phi}_{12}\boldsymbol{\Upsilon}_{12}^T + \boldsymbol{\Upsilon}_{12}\boldsymbol{\Phi}_{12}^T\right)[S_{t2}s_{21}(\dot\theta_1 - \dot\theta_2) + S_3s_{31}(\dot\theta_1 - \dot\theta_3)]$$
$$+ 2\boldsymbol{\Upsilon}_{12}\boldsymbol{\Upsilon}_{12}^T S_3L_2s_{32}(\dot\theta_2 - \dot\theta_3)$$

$$G_{78} = -2\left(\boldsymbol{\Phi}_{12}\bar{\boldsymbol{\Phi}}_{t2}^T s_{21}\dot\theta_2 + S_3\boldsymbol{\Phi}_{12}\boldsymbol{\Upsilon}_{23}^T s_{31}\dot\theta_3\right.$$
$$\left. + S_3\boldsymbol{\Upsilon}_{12}\boldsymbol{\Phi}_{23}^T s_{32}\dot\theta_2 + S_3L_2\boldsymbol{\Upsilon}_{12}\boldsymbol{\Upsilon}_{23}^T s_{32}\dot\theta_3\right)$$

$$\boldsymbol{G}_{83} = 2\bar{\boldsymbol{\Phi}}_{t2}L_0s_{20}\dot\theta_0 + 2S_3\boldsymbol{\Upsilon}_{23}L_0s_{30}\dot\theta_0$$

$$\boldsymbol{G}_{84} = 2\bar{\boldsymbol{\Phi}}_{t2}L_1s_{21}\dot\theta_1 + 2S_3\boldsymbol{\Upsilon}_{23}L_1s_{31}\dot\theta_1$$

$$\boldsymbol{G}_{85} = 2S_3\boldsymbol{\Upsilon}_{23}L_2s_{32}\dot\theta_3$$

$$\boldsymbol{G}_{86} = -2S_3\boldsymbol{\Phi}_{23}s_{32}\dot\theta_3$$

$$G_{87} = 2\left(\bar{\boldsymbol{\Phi}}_{t2}\boldsymbol{\Phi}_{12}^T s_{21}\dot\theta_1 + S_3\boldsymbol{\Upsilon}_{23}\boldsymbol{\Phi}_{12}^T s_{31}\dot\theta_1\right.$$
$$\left. - S_3\boldsymbol{\Phi}_{23}\boldsymbol{\Upsilon}_{12}^T s_{32}\dot\theta_3 + S_3L_2\boldsymbol{\Upsilon}_{23}\boldsymbol{\Upsilon}_{12}^T s_{32}\dot\theta_2\right)$$

$$G_{88} = \left(\boldsymbol{\Phi}_{23}\boldsymbol{\Upsilon}_{23}^T + \boldsymbol{\Upsilon}_{23}\boldsymbol{\Phi}_{23}^T\right)S_3s_{32}(\dot\theta_2 - \dot\theta_3)$$

and

$$C = \begin{bmatrix} 0 & 0 & C_{13} & C_{14} & C_{15} & C_{16} & \boldsymbol{C}_{17} & \boldsymbol{C}_{18} \\ 0 & 0 & C_{23} & C_{24} & C_{25} & C_{26} & \boldsymbol{C}_{27} & \boldsymbol{C}_{28} \\ 0 & 0 & C_{33} & C_{34} & C_{35} & C_{36} & \boldsymbol{C}_{37} & \boldsymbol{C}_{38} \\ 0 & 0 & C_{43} & C_{44} & C_{45} & C_{46} & \boldsymbol{C}_{47} & \boldsymbol{C}_{48} \\ 0 & 0 & C_{53} & C_{54} & C_{55} & C_{56} & \boldsymbol{C}_{57} & \boldsymbol{C}_{58} \\ 0 & 0 & C_{63} & C_{64} & C_{65} & C_{66} & \boldsymbol{C}_{67} & \boldsymbol{C}_{68} \\ 0 & 0 & \boldsymbol{C}_{73} & \boldsymbol{C}_{74} & \boldsymbol{C}_{75} & \boldsymbol{C}_{76} & C_{77} + K_1 & C_{78} \\ 0 & 0 & \boldsymbol{C}_{83} & \boldsymbol{C}_{84} & \boldsymbol{C}_{85} & \boldsymbol{C}_{86} & C_{87} & C_{88} + K_2 \end{bmatrix} \tag{A6}$$

where

$$C_{13} = -S_{ty}\ddot\theta_0 + S_{tx}\dot\theta_0^2, \qquad C_{14} = -S_{t1}c_1\ddot\theta_1 + S_{t1}s_1\dot\theta_1^2$$

$$C_{15} = -S_{t2}c_2\ddot\theta_2 + S_{t2}s_2\dot\theta_2^2$$

$$C_{16} = -S_3c_3\ddot\theta_3 + S_3s_3\dot\theta_3^2$$

$$\boldsymbol{C}_{17} = -\bar{\boldsymbol{\Phi}}_{t1}^T c_1\ddot\theta_1 - S_{t2}\boldsymbol{\Upsilon}_{12}^T c_2\ddot\theta_2 + \bar{\boldsymbol{\Phi}}_{t1}^T s_1\dot\theta_1^2 + S_{t2}\boldsymbol{\Upsilon}_{12}^T s_2\dot\theta_2^2$$
$$- S_3\boldsymbol{\Upsilon}_{12}^T c_3\ddot\theta_3 + S_3\boldsymbol{\Upsilon}_{12}^T s_3\dot\theta_3^2$$

$$\boldsymbol{C}_{18} = -\bar{\boldsymbol{\Phi}}_{t2}^T c_2\ddot\theta_2 - S_3\boldsymbol{\Upsilon}_{23}^T c_3\ddot\theta_3 + \bar{\boldsymbol{\Phi}}_{t2}^T s_2\dot\theta_2^2 + S_3\boldsymbol{\Upsilon}_{23}^T s_3\dot\theta_3^2$$

$$C_{23} = -S_{tx}\ddot\theta_0 - S_{ty}\dot\theta_0^2$$

$$C_{24} = -S_{t1}s_1\ddot\theta_1 - S_{t1}c_1\dot\theta_1^2$$

$$C_{25} = -S_{t2}s_2\ddot\theta_2 - S_{t2}c_2\dot\theta_2^2$$

$$C_{26} = -S_3s_3\ddot\theta_3 - S_3c_3\dot\theta_3^2$$

$$\boldsymbol{C}_{27} = -\bar{\boldsymbol{\Phi}}_{t1}^T s_1\ddot\theta_1 - S_{t2}\boldsymbol{\Upsilon}_{12}^T s_2\ddot\theta_2 - \bar{\boldsymbol{\Phi}}_{t1}^T c_1\dot\theta_1^2 - S_{t2}\boldsymbol{\Upsilon}_{12}^T c_2\dot\theta_2^2$$
$$- S_3\boldsymbol{\Upsilon}_{12}^T s_3\ddot\theta_3 - S_3\boldsymbol{\Upsilon}_{12}^T c_3\dot\theta_3^2$$

$$C_{28} = -\bar{\boldsymbol{\Phi}}_{t2}^T s_2\ddot{\theta}_2 - S_3\boldsymbol{\Upsilon}_{23}^T s_3\ddot{\theta}_3 - \bar{\boldsymbol{\Phi}}_{t2}^T c_2\dot{\theta}_2^2 - S_3\boldsymbol{\Upsilon}_{23}^T c_3\dot{\theta}_3^2$$

$$C_{33} = -S_{ty}\ddot{x}_0 - S_{tx}\ddot{y}_0 - S_{t1}L_0c_{10}\ddot{\theta}_1 - S_{t2}L_0c_{20}\ddot{\theta}_2 - S_3L_0c_{30}\ddot{\theta}_3 + S_{t1}L_0s_{10}\dot{\theta}_1^2 + S_{t2}L_0s_{20}\dot{\theta}_2^2 + S_3L_0s_{30}\dot{\theta}_3^2$$

$$C_{34} = S_{t1}L_0c_{10}\ddot{\theta}_1 - S_{t1}L_0s_{10}\dot{\theta}_1^2$$

$$C_{35} = S_{t2}L_0c_{20}\ddot{\theta}_2 - S_{t2}L_0s_{20}\dot{\theta}_2^2$$

$$C_{36} = S_3L_0c_{30}\ddot{\theta}_3 - S_3L_0s_{30}\dot{\theta}_3^2$$

$$C_{37} = \bar{\boldsymbol{\Phi}}_{t1}^T L_0c_{10}\ddot{\theta}_1 + S_{t2}\boldsymbol{\Upsilon}_{12}^T L_0c_{20}\ddot{\theta}_2 - \bar{\boldsymbol{\Phi}}_{t1}^T L_0s_{10}\dot{\theta}_1^2 - S_{t2}\boldsymbol{\Upsilon}_{12}^T L_0s_{20}\dot{\theta}_2^2 + S_3L_0\boldsymbol{\Upsilon}_{12}^T c_{30}\ddot{\theta}_3 - S_3L_0\boldsymbol{\Upsilon}_{12}^T s_{30}\dot{\theta}_3^2$$

$$C_{38} = \bar{\boldsymbol{\Phi}}_{t2}^T L_0c_{20}\ddot{\theta}_2 + S_3\boldsymbol{\Upsilon}_{23}^T L_0c_{30}\ddot{\theta}_3 - \bar{\boldsymbol{\Phi}}_{t2}^T L_0s_{20}\dot{\theta}_2^2 - S_3\boldsymbol{\Upsilon}_{23}^T L_0s_{30}\dot{\theta}_3^2$$

$$C_{43} = -S_{t1}L_0c_{10}\ddot{\theta}_0 - S_{t1}L_0s_{10}\dot{\theta}_0^2$$

$$C_{44} = -S_{t1}c_1\ddot{x}_0 - S_{t1}s_1\ddot{y}_0 + S_{t1}L_0c_{10}\ddot{\theta}_0 + S_{t2}L_1s_{21}\ddot{\theta}_2 + S_3L_1s_{31}\ddot{\theta}_3 + S_{t1}L_0s_{10}\dot{\theta}_0^2 + S_{t2}L_1c_{21}\dot{\theta}_2^2 + S_3L_1c_{31}\dot{\theta}_3^2$$

$$C_{45} = -S_{t2}L_1s_{21}\ddot{\theta}_2 - S_{t2}L_1c_{21}\dot{\theta}_2^2$$

$$C_{46} = -S_3L_1s_{31}\ddot{\theta}_3 - S_3L_1c_{31}\dot{\theta}_3^2$$

$$C_{47} = -\bar{\boldsymbol{\Phi}}_{t1}^T c_1\ddot{x}_0 - \bar{\boldsymbol{\Phi}}_{t1}^T s_1\ddot{y}_0 + \bar{\boldsymbol{\Phi}}_{t1}^T L_0c_{10}\ddot{\theta}_0 - S_{t2}L_1\boldsymbol{\Upsilon}_{12}^T s_{21}\ddot{\theta}_2 + S_{t2}\boldsymbol{\Phi}_{12}^T s_{21}\ddot{\theta}_2 + S_3\boldsymbol{\Phi}_{12}^T s_{31}\ddot{\theta}_3 + \bar{\boldsymbol{\Phi}}_{t1}^T L_0s_{10}\dot{\theta}_0^2 - S_{t2}L_1\boldsymbol{\Upsilon}_{12}^T c_{21}\dot{\theta}_2^2 + S_{t2}\boldsymbol{\Phi}_{12}^T c_{21}\dot{\theta}_2^2 + S_3\boldsymbol{\Phi}_{12}^T c_{31}\dot{\theta}_3^2 - S_3L_1\boldsymbol{\Upsilon}_{12}^T s_{31}\ddot{\theta}_3 - S_3L_1\boldsymbol{\Upsilon}_{12}^T c_{31}\dot{\theta}_3^2$$

$$C_{48} = -\bar{\boldsymbol{\Phi}}_{t2}^T L_1s_{21}\ddot{\theta}_2 - S_3L_1\boldsymbol{\Upsilon}_{23}^T s_{31}\ddot{\theta}_3 - \bar{\boldsymbol{\Phi}}_{t2}^T L_1c_{21}\dot{\theta}_2^2 - S_3L_1\boldsymbol{\Upsilon}_{23}^T c_{31}\dot{\theta}_3^2$$

$$C_{53} = -S_{t2}L_0c_{20}\ddot{\theta}_0 - S_{t2}L_0s_{20}\dot{\theta}_0^2$$

$$C_{54} = S_{t2}L_1s_{21}\ddot{\theta}_1 - S_{t2}L_1c_{21}\dot{\theta}_1^2$$

$$C_{55} = -S_{t2}c_2\ddot{x}_0 - S_{t2}s_2\ddot{y}_0 + S_{t2}L_0c_{20}\ddot{\theta}_0 - S_{t2}L_1s_{21}\ddot{\theta}_1 + S_3L_2s_{32}\ddot{\theta}_3 + S_{t2}L_0s_{20}\dot{\theta}_0^2 + S_{t2}L_1c_{21}\dot{\theta}_1^2 + S_3L_2c_{32}\dot{\theta}_3^2$$

$$C_{56} = -S_3L_2s_{32}\ddot{\theta}_3 - S_3L_2c_{32}\dot{\theta}_3^2$$

$$C_{57} = -S_{t2}\boldsymbol{\Upsilon}_{12}^T c_2\ddot{x}_0 - S_{t2}\boldsymbol{\Upsilon}_{12}^T s_2\ddot{y}_0 + S_{t2}L_0\boldsymbol{\Upsilon}_{12}^T c_{20}\ddot{\theta}_0 + S_{t2}\boldsymbol{\Phi}_{12}^T s_{21}\ddot{\theta}_1 - S_{t2}L_1\boldsymbol{\Upsilon}_{12}^T s_{21}\ddot{\theta}_1 + S_{t2}L_0\boldsymbol{\Upsilon}_{12}^T s_{20}\dot{\theta}_0^2 - S_{t2}\boldsymbol{\Phi}_{12}^T c_{21}\dot{\theta}_1^2 + S_{t2}L_1\boldsymbol{\Upsilon}_{12}^T c_{21}\dot{\theta}_1^2$$

$$C_{58} = -\bar{\boldsymbol{\Phi}}_{t2}^T c_2\ddot{x}_0 - \bar{\boldsymbol{\Phi}}_{t2}^T s_2\ddot{y}_0 + \bar{\boldsymbol{\Phi}}_{t2}^T L_0c_{20}\ddot{\theta}_0 - \bar{\boldsymbol{\Phi}}_{t2}^T L_1s_{21}\ddot{\theta}_1 + S_3\boldsymbol{\Phi}_{23}^T s_{32}\ddot{\theta}_3 - S_3L_2\boldsymbol{\Upsilon}_{23}^T s_{32}\ddot{\theta}_3 + \bar{\boldsymbol{\Phi}}_{t2}^T L_0s_{20}\dot{\theta}_0^2 + \bar{\boldsymbol{\Phi}}_{t2}^T L_1c_{21}\dot{\theta}_1^2 + S_3\boldsymbol{\Phi}_{23}^T c_{32}\dot{\theta}_3^2 - S_3L_2\boldsymbol{\Upsilon}_{23}^T c_{32}\dot{\theta}_3^2$$

$$C_{63} = -S_3L_0c_{30}\ddot{\theta}_0 - S_3L_0s_{30}\dot{\theta}_0^2$$

$$C_{64} = S_3L_1s_{31}\ddot{\theta}_1 - S_3L_1c_{31}\dot{\theta}_1^2$$

$$C_{65} = S_3L_2s_{32}\ddot{\theta}_2 - S_3L_2c_{32}\dot{\theta}_2^2$$

$$C_{66} = -S_3c_3\ddot{x}_0 - S_3s_3\ddot{y}_0 + S_3L_0c_{30}\ddot{\theta}_0 - S_3L_1s_{31}\ddot{\theta}_1 - S_3L_2s_{32}\ddot{\theta}_2 + S_3L_0s_{30}\dot{\theta}_0^2 + S_3L_1c_{31}\dot{\theta}_1^2 + S_3L_2c_{32}\dot{\theta}_2^2$$

$$C_{67} = S_3\boldsymbol{\Phi}_{12}^T s_{31}\ddot{\theta}_1 - S_3\boldsymbol{\Phi}_{12}^T c_{31}\dot{\theta}_1^2 - S_3\boldsymbol{\Upsilon}_{12}^T c_3\ddot{x}_0 - S_3\boldsymbol{\Upsilon}_{12}^T s_3\ddot{y}_0 + S_3\boldsymbol{\Upsilon}_{12}^T L_0c_{30}\ddot{\theta}_0 - S_3\boldsymbol{\Upsilon}_{12}^T L_1s_{31}\ddot{\theta}_1 + S_3\boldsymbol{\Upsilon}_{12}^T L_0s_{30}\dot{\theta}_0^2 + S_3\boldsymbol{\Upsilon}_{12}^T L_1c_{31}\dot{\theta}_1^2 \tag{A7}$$

$$C_{68} = -S_3\boldsymbol{\Upsilon}_{23}^T c_3\ddot{x}_0 - S_3\boldsymbol{\Upsilon}_{23}^T s_3\ddot{y}_0 + S_3\boldsymbol{\Upsilon}_{23}^T L_0c_{30}\ddot{\theta}_0 - S_3\boldsymbol{\Upsilon}_{23}^T L_1s_{31}\ddot{\theta}_1 - S_3\boldsymbol{\Upsilon}_{23}^T L_2s_{32}\ddot{\theta}_2 + S_3\boldsymbol{\Phi}_{23}^T s_{32}\ddot{\theta}_2 + S_3\boldsymbol{\Upsilon}_{23}^T L_0s_{30}\dot{\theta}_0^2 + S_3\boldsymbol{\Upsilon}_{23}^T L_1c_{31}\dot{\theta}_1^2 + S_3\boldsymbol{\Upsilon}_{23}^T L_2c_{32}\dot{\theta}_2^2 - S_3\boldsymbol{\Phi}_{23}^T c_{32}\dot{\theta}_2^2$$

$$C_{73} = -\bar{\boldsymbol{\Phi}}_{t1}L_0c_{10}\ddot{\theta}_0 - S_{t2}\boldsymbol{\Upsilon}_{12}L_0c_{20}\ddot{\theta}_0 - \bar{\boldsymbol{\Phi}}_{t1}L_0s_{10}\dot{\theta}_0^2 - S_{t2}\boldsymbol{\Upsilon}_{12}L_0s_{20}\dot{\theta}_0^2 - S_3\boldsymbol{\Upsilon}_{12}L_0c_{30}\ddot{\theta}_0 - S_3\boldsymbol{\Upsilon}_{12}L_0s_{30}\dot{\theta}_0^2$$

$$C_{74} = -\bar{\boldsymbol{\Phi}}_{t1}c_1\ddot{x}_0 - \bar{\boldsymbol{\Phi}}_{t1}s_1\ddot{y}_0 + \bar{\boldsymbol{\Phi}}_{t1}L_0c_{10}\ddot{\theta}_0 + S_{t2}L_1\boldsymbol{\Upsilon}_{12}s_{21}\ddot{\theta}_1 + S_{t2}\boldsymbol{\Phi}_{12}s_{21}\ddot{\theta}_2 + S_3\boldsymbol{\Phi}_{12}s_{31}\ddot{\theta}_3 + \bar{\boldsymbol{\Phi}}_{t1}L_0s_{10}\dot{\theta}_0^2 - S_{t2}L_1\boldsymbol{\Upsilon}_{12}c_{21}\dot{\theta}_1^2 + S_{t2}\boldsymbol{\Phi}_{12}c_{21}\dot{\theta}_2^2 + S_3\boldsymbol{\Phi}_{12}c_{31}\dot{\theta}_3^2 + S_3\boldsymbol{\Upsilon}_{12}L_1s_{31}\ddot{\theta}_1 - S_3\boldsymbol{\Upsilon}_{12}L_1c_{31}\dot{\theta}_1^2$$

$$C_{75} = -S_{t2}\boldsymbol{\Upsilon}_{12}c_2\ddot{x}_0 - S_{t2}\boldsymbol{\Upsilon}_{12}s_2\ddot{y}_0 + S_{t2}L_0\boldsymbol{\Upsilon}_{12}c_{20}\ddot{\theta}_0 - S_{t2}\boldsymbol{\Phi}_{12}s_{21}\ddot{\theta}_2 - S_{t2}L_1\boldsymbol{\Upsilon}_{12}s_{21}\ddot{\theta}_1 + S_3L_2\boldsymbol{\Upsilon}_{12}s_{32}\ddot{\theta}_3 + S_{t2}L_0\boldsymbol{\Upsilon}_{12}s_{20}\dot{\theta}_0^2 - S_{t2}\boldsymbol{\Phi}_{12}c_{21}\dot{\theta}_2^2 + S_{t2}L_1\boldsymbol{\Upsilon}_{12}c_{21}\dot{\theta}_1^2 + S_3L_2\boldsymbol{\Upsilon}_{12}c_{32}\dot{\theta}_3^2 + S_3\boldsymbol{\Upsilon}_{12}L_2s_{32}\ddot{\theta}_2 - S_3\boldsymbol{\Upsilon}_{12}L_2c_{32}\dot{\theta}_2^2$$

$$C_{76} = -S_3\boldsymbol{\Phi}_{12}s_{31}\ddot{\theta}_3 - S_3\boldsymbol{\Upsilon}_{12}L_2s_{32}\ddot{\theta}_3 - S_3\boldsymbol{\Phi}_{12}c_{31}\dot{\theta}_3^2 - S_3\boldsymbol{\Upsilon}_{12}L_2c_{32}\dot{\theta}_3^2 - S_3\boldsymbol{\Upsilon}_{12}c_3\ddot{x}_0 - S_3\boldsymbol{\Upsilon}_{12}s_3\ddot{y}_0 + S_3\boldsymbol{\Upsilon}_{12}L_0c_{30}\ddot{\theta}_0 - S_3\boldsymbol{\Upsilon}_{12}L_1s_{31}\ddot{\theta}_1 - S_3\boldsymbol{\Upsilon}_{12}L_2s_{32}\ddot{\theta}_2 + S_3\boldsymbol{\Upsilon}_{12}L_0s_{30}\dot{\theta}_0^2 + S_3\boldsymbol{\Upsilon}_{12}L_1c_{31}\dot{\theta}_1^2 + S_3\boldsymbol{\Upsilon}_{12}L_2c_{32}\dot{\theta}_2^2$$

$$C_{77} = -S_{t2}\boldsymbol{\Upsilon}_{12}\boldsymbol{\Upsilon}_{12}^T\big(c_2\ddot{x}_0 + s_2\ddot{y}_0 - L_0c_{20}\ddot{\theta}_0 + L_1s_{21}\ddot{\theta}_1 - L_0s_{20}\dot{\theta}_0^2 - L_1c_{21}\dot{\theta}_1^2\big) - S_3\boldsymbol{\Upsilon}_{12}\boldsymbol{\Upsilon}_{12}^T\big(c_3\ddot{x}_0 + s_3\ddot{y}_0 - L_0c_{30}\ddot{\theta}_0 + L_1s_{31}\ddot{\theta}_1 - L_0s_{30}\dot{\theta}_0^2 - L_1c_{31}\dot{\theta}_1^2\big) + S_{t2}\boldsymbol{\Upsilon}_{12}\boldsymbol{\Phi}_{12}^T\big[s_{21}(\ddot{\theta}_1 - \ddot{\theta}_2) - c_{21}\big(\dot{\theta}_1^2 + \dot{\theta}_2^2\big)\big] + S_3\boldsymbol{\Upsilon}_{12}\boldsymbol{\Phi}_{12}^T\big[s_{31}(\ddot{\theta}_1 - \ddot{\theta}_3) - c_{31}\big(\dot{\theta}_1^2 + \dot{\theta}_3^2\big)\big] - \big[\boldsymbol{\Lambda}_1 + (m_2 + m_3)\boldsymbol{\Phi}_{12}\boldsymbol{\Phi}_{12}^T\big]\dot{\theta}_1^2$$

$$C_{78} = -\boldsymbol{\Upsilon}_{12}\bar{\boldsymbol{\Phi}}_{t2}^T\big(c_2\ddot{x}_0 + s_2\ddot{y}_0 - L_0c_{20}\ddot{\theta}_0 + L_1s_{21}\ddot{\theta}_1 - L_0s_{20}\dot{\theta}_0^2 - L_1c_{21}\dot{\theta}_1^2\big) - S_3\boldsymbol{\Upsilon}_{12}\boldsymbol{\Upsilon}_{23}^T\big(c_3\ddot{x}_0 + s_3\ddot{y}_0 - L_0c_{30}\ddot{\theta}_0 + L_1s_{31}\ddot{\theta}_1 + L_2s_{32}\ddot{\theta}_2 - L_0s_{30}\dot{\theta}_0^2 - L_1c_{31}\dot{\theta}_1^2 - L_2c_{32}\dot{\theta}_2^2\big) - \boldsymbol{\Phi}_{12}\bar{\boldsymbol{\Phi}}_{t2}^T\big(s_{21}\ddot{\theta}_2 + c_{21}\dot{\theta}_2^2\big) - S_3\boldsymbol{\Phi}_{12}\boldsymbol{\Upsilon}_{23}^T\big(s_{31}\ddot{\theta}_3 + c_{31}\dot{\theta}_3^2\big) + S_3\boldsymbol{\Upsilon}_{12}\boldsymbol{\Phi}_{23}^T\big(s_{32}\ddot{\theta}_3 + c_{32}\dot{\theta}_3^2\big) + S_3\boldsymbol{\Upsilon}_{12}\boldsymbol{\Phi}_{23}^T\big(s_{32}\ddot{\theta}_2 - c_{32}\dot{\theta}_2^2\big) - S_3\boldsymbol{\Upsilon}_{12}\boldsymbol{\Upsilon}_{23}^T L_2\big(s_{32}\ddot{\theta}_3 + c_{32}\dot{\theta}_3^2\big)$$

$$C_{83} = \bar{\boldsymbol{\Phi}}_{t2}L_0s_{20}\ddot{\theta}_0 + S_3\boldsymbol{\Upsilon}_{23}L_0s_{30}\ddot{\theta}_0 - \bar{\boldsymbol{\Phi}}_{t2}L_0c_{20}\dot{\theta}_0^2 - S_3\boldsymbol{\Upsilon}_{23}L_0c_{30}\dot{\theta}_0^2$$

$$C_{84} = \bar{\mathbf{\Phi}}_{t2} L_1 s_{21} \ddot{\theta}_1 + S_3 L_1 \mathbf{\Upsilon}_{23} s_{31} \ddot{\theta}_1 - \bar{\mathbf{\Phi}}_{t2} L_1 c_{21} \dot{\theta}_1^2 - S_3 L_1 \mathbf{\Upsilon}_{23} c_{31} \dot{\theta}_1^2$$

$$C_{85} = -\bar{\mathbf{\Phi}}_{t2} c_2 \ddot{x}_0 - \bar{\mathbf{\Phi}}_{t2} s_2 \ddot{y}_0 - \bar{\mathbf{\Phi}}_{t2} L_0 s_{20} \ddot{\theta}_0 - \bar{\mathbf{\Phi}}_{t2} L_1 s_{21} \ddot{\theta}_1 + S_3 L_2 \mathbf{\Upsilon}_{23} s_{32} \ddot{\theta}_2 + S_3 \mathbf{\Phi}_{23} s_{32} \ddot{\theta}_3 + \bar{\mathbf{\Phi}}_{t2} L_0 c_{20} \dot{\theta}_0^2 + \bar{\mathbf{\Phi}}_{t2} L_1 c_{21} \dot{\theta}_1^2 + S_3 \mathbf{\Phi}_{23} c_{32} \dot{\theta}_3^2 - S_3 L_2 \mathbf{\Upsilon}_{23} c_{32} \dot{\theta}_2^2$$

$$C_{86} = -S_3 \mathbf{\Upsilon}_{23} c_3 \ddot{x}_0 - S_3 \mathbf{\Upsilon}_{23} s_3 \ddot{y}_0 - S_3 \mathbf{\Upsilon}_{23} L_0 s_{30} \ddot{\theta}_0 - S_3 \mathbf{\Upsilon}_{23} L_1 s_{31} \ddot{\theta}_1 - S_3 \mathbf{\Upsilon}_{23} L_2 s_{32} \ddot{\theta}_2 - S_3 \mathbf{\Phi}_{23} s_{32} \ddot{\theta}_3 + S_3 \mathbf{\Upsilon}_{23} L_0 c_{30} \dot{\theta}_0^2 + S_3 \mathbf{\Upsilon}_{23} L_1 c_{31} \dot{\theta}_1^2 + S_3 \mathbf{\Upsilon}_{23} L_2 c_{32} \dot{\theta}_2^2 - S_3 \mathbf{\Phi}_{23} c_{32} \dot{\theta}_3^2$$

$$C_{87} = -\bar{\mathbf{\Phi}}_{t2} \mathbf{\Upsilon}_{12}^T \big(c_2 \ddot{x}_0 + s_2 \ddot{y}_0 + L_0 s_{20} \ddot{\theta}_0 + L_1 s_{21} \ddot{\theta}_1 - L_0 c_{20} \dot{\theta}_0^2 - L_1 c_{21} \dot{\theta}_1^2 \big) - S_3 \mathbf{\Upsilon}_{23} \mathbf{\Upsilon}_{12}^T \big(c_3 \ddot{x}_0 + s_3 \ddot{y}_0 + L_0 s_{30} \ddot{\theta}_0 + L_1 s_{31} \ddot{\theta}_1 - L_0 c_{30} \dot{\theta}_0^2 - L_1 c_{31} \dot{\theta}_1^2 \big) + \bar{\mathbf{\Phi}}_{t2} \mathbf{\Phi}_{12}^T \big(s_{21} \ddot{\theta}_1 - c_{21} \dot{\theta}_1^2 \big) + S_3 \mathbf{\Upsilon}_{23} \mathbf{\Phi}_{12}^T \big(s_{31} \ddot{\theta}_1 - c_{31} \dot{\theta}_1^2 \big)$$

$$C_{88} = -S_3 \mathbf{\Upsilon}_{23} \mathbf{\Upsilon}_{23}^T \big(c_3 \ddot{x}_0 + s_3 \ddot{y}_0 + L_0 s_{30} \ddot{\theta}_0 + L_1 s_{31} \ddot{\theta}_1 + L_2 s_{32} \ddot{\theta}_2 - L_0 c_{30} \dot{\theta}_0^2 - L_1 c_{31} \dot{\theta}_1^2 - L_2 c_{32} \dot{\theta}_2^2 \big) + S_3 \mathbf{\Upsilon}_{23} \mathbf{\Phi}_{23}^T \big[s_{32} \big(\ddot{\theta}_2 - \ddot{\theta}_3 \big) - c_{32} \big(\dot{\theta}_2^2 + \dot{\theta}_3^2 \big) \big] - \big[\mathbf{\Lambda}_2 + m_3 \mathbf{\Phi}_{23} \mathbf{\Phi}_{23}^T \big] \dot{\theta}_2^2$$

The disturbance vector $\boldsymbol{d}$ in Eq. (48) is defined as

$$\boldsymbol{d} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{d}_7^T & \boldsymbol{d}_8^T \end{bmatrix}^T \tag{A8}$$

where

$$\boldsymbol{d}_7 = [\tilde{\mathbf{\Phi}}_1 + (m_2 + m_3) L_1 \mathbf{\Phi}_{12}] \ddot{\theta}_1 + \bar{\mathbf{\Phi}}_{t1} \big(-s_1 \ddot{x}_0 + c_1 \ddot{y}_0 + L_0 s_{10} \ddot{\theta}_0 - L_0 c_{10} \dot{\theta}_0^2 \big) + S_{t2} \mathbf{\Upsilon}_{12} \big(-s_2 \ddot{x}_0 + c_2 \ddot{y}_0 + L_0 s_{20} \ddot{\theta}_0 + L_1 c_{21} \ddot{\theta}_1 - L_0 c_{20} \dot{\theta}_0^2 + L_1 s_{21} \dot{\theta}_1^2 \big) + I_{t2} \mathbf{\Upsilon}_{12} \ddot{\theta}_2 + S_{t2} \mathbf{\Phi}_{12} \big(c_{21} \ddot{\theta}_2 - s_{21} \dot{\theta}_2^2 \big) + S_3 \mathbf{\Phi}_{12} \big(c_{31} \ddot{\theta}_3 - s_{31} \dot{\theta}_3^2 \big) + S_3 \mathbf{\Upsilon}_{12} L_2 \big(c_{32} \ddot{\theta}_3 - s_{32} \dot{\theta}_3^2 \big) + I_3 \mathbf{\Upsilon}_{12} \ddot{\theta}_3 + S_3 \mathbf{\Upsilon}_{12} \big(-s_3 \ddot{x}_0 + c_3 \ddot{y}_0 + L_0 s_{30} \ddot{\theta}_0 + L_1 c_{31} \ddot{\theta}_1 + L_2 c_{32} \ddot{\theta}_2 - L_0 c_{30} \dot{\theta}_0^2 + L_1 s_{31} \dot{\theta}_1^2 + L_2 s_{32} \dot{\theta}_2^2 \big) \tag{A9}$$

$$\boldsymbol{d}_8 = [\tilde{\mathbf{\Phi}}_2 + m_3 L_2 \mathbf{\Phi}_{23}] \ddot{\theta}_2 + \bar{\mathbf{\Phi}}_{t2} \big(-s_2 \ddot{x}_0 + c_2 \ddot{y}_0 + L_0 c_{20} \ddot{\theta}_0 + L_0 s_{20} \dot{\theta}_0^2 + L_1 c_{21} \ddot{\theta}_1 + L_1 s_{21} \dot{\theta}_1^2 \big) + S_3 \mathbf{\Upsilon}_{23} \big(-s_3 \ddot{x}_0 + c_3 \ddot{y}_0 + L_0 c_{30} \ddot{\theta}_0 + L_1 c_{31} \ddot{\theta}_1 + L_2 c_{32} \ddot{\theta}_2 + L_0 s_{30} \dot{\theta}_0^2 + L_1 s_{31} \dot{\theta}_1^2 + L_2 s_{32} \dot{\theta}_2^2 \big) + I_3 \mathbf{\Upsilon}_{23} \ddot{\theta}_3 + S_3 \mathbf{\Phi}_{23} \big(c_{32} \ddot{\theta}_3 - s_{32} \dot{\theta}_3^2 \big)$$

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References

[1]Wang, P. K. C., "Automatic Assembly of Space Station," *Proceedings of the Workshop on Identification and Control of Flexible Space Structures* (San Diego, CA), Vol. 1, 1985, pp. 67–101.

[2]Lindberg, R. E., Longman, R. W., and Zedd, M. F., "Kinematic and Dynamic Properties of an Elbow Manipulator Mounted on a Satellite," *Journal of the Astronautical Sciences*, Vol. 38, No. 4, 1990, pp. 397–421.

[3]Longman, R. W., "The Kinetics and Workspace of a Satellite-Mounted Robot," *Journal of the Astronautical Sciences*, Vol. 38, No. 4, 1990, pp. 423–439.

[4]Meirovitch, L., and Kwak, M. K., "Dynamics and Control of Spacecraft with Retargeting Flexible Antennas," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 2, 1990, pp. 241–248.

[5]Vafa, Z., and Dubowsky, S., "On the Dynamics of Space Manipulators Using the Virtual Manipulator, with Applications to Path Planning," *Journal of the Astronautical Sciences*, Vol. 38, No. 4, 1990, pp. 441–472.

[6]Alexander, H. L., and Cannon, R. H., "An Extended Operational-Space Control Algorithm for Satellite Manipulators," *Journal of the Astronautical Sciences*, Vol. 38, No. 4, 1990, pp. 473–486.

[7]Nenchev, D., Umetani, Y., and Yoshida, K., "Analysis of a Redundant Free-Flying Spacecraft/Manipulator System," *IEEE Transactions on Robotics and Automation*, Vol. 8, No. 1, 1992, pp. 1–6.

[8]Nakamura, Y., and Mukherjee, R., "Nonholonomic Path Planning via a Bidirectional Approach," *IEEE Transactions on Robotics and Automation*, Vol. 7, No. 4, 1991, pp. 500–514.

[9]Nakamura, Y., *Advanced Robotics: Redundancy and Optimization*, Addison–Wesley, Reading, MA, 1991.

[10]De Silva, C. W., "Trajectory Design for Robotic Manipulators in Space Applications," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 3, 1991, pp. 670–674.

[11]Book, W. J., "Structural Flexibility of Motion Systems in the Space Environment," *IEEE Transactions on Robotics and Automation*, Vol. 9, No. 5, 1993, pp. 524–530.

[12]Koivo, A. J., and Arnautovic, S. H., "Dynamic Optimum Control of Redundant Manipulators," *Proceedings of the 1991 IEEE International Conference on Robotics and Automation* (Sacramento, CA), Inst. for Electrical and Electronics Engineers, 1991, pp. 466–471.

[13]Watson, L. T., "Globally Convergent Homotopy Algorithms for Nonlinear Systems of Equations," *Nonlinear Dynamics*, Vol. 1, 1990, pp. 143–191.

[14]Kazerounian, K., and Wang, Z., "Global Versus Local Optimization in Redundancy Resolution of Robotic Manipulators," *International Journal of Robotics Research*, Vol. 7, No. 5, 1988, pp. 3–12.

[15]Meirovitch, L., and Kwak, M. K., "Rayleigh-Ritz Based Substructure Synthesis for Flexible Multibody Systems," *AIAA Journal*, Vol. 19, No. 10, 1991, pp. 1709–1719.

[16]Ascher, U., Mattheij, R., and Russell, R., *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, Prentice–Hall, Englewood Cliffs, NJ, 1988.

[17]Meirovitch, L., *Dynamics and Control of Structures*, Wiley-Interscience, New York, 1990.

[18]Kenney, R. S., and Leipnik, R. B., "Numerical Integration of the Differential Matrix Riccati Equation," *IEEE Transactions on Automatic Control*, Vol. 30, No. 10, 1990, pp. 962–970.